The longest excursion of stochastic processes in nonequilibrium systems

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We consider the excursions, i.e. the intervals between consecutive zeros, of stochastic processes that arise in a variety of nonequilibrium systems and study the temporal growth of the *longest* one $l_{\max}(t)$ up to time t. For smooth processes, we find a universal linear growth $\langle l_{\max}(t)\rangle \simeq Q_\infty\, t$ with a model dependent amplitude Q_∞. In contrast, for non-smooth processes with a persistence exponent θ, we show that $\langle l_{\max}(t)\rangle$ has a linear growth if $\theta < \theta_c$ while $\langle l_{\max}(t)\rangle \sim t^{1-\psi}$ if $\theta > \theta_c$. The amplitude Q_∞ and the exponent ψ are novel quantities associated to nonequilibrium dynamics. These behaviors are obtained by exact analytical calculations for renewal and multiplicative processes and numerical simulations for other systems such as the coarsening dynamics in Ising model as well as the diffusion equation with random initial conditions.

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Introduction. Nonequilibrium dynamics in many-body systems keep offering new challenges despite several decades of research. An example of such a system, among others, is the Ising model undergoing phase ordering after a rapid quench in temperature [1]. In such systems, the relevant stochastic process $X(t)$ that represents, at a fixed point in space, the evolving spin in the Ising model (or e.g., the field in the diffusion equation) is generically a complex one with nontrivial history dependence. Traditional two-time correlation function $\langle X(t_1)X(t_2)\rangle$ is typically not sufficient to characterize the complex temporal history of such a process. One simple measure of this history dependence that has attracted much attention in the recent past, both theoretically [2, 3] and experimentally [4], is the persistence $p(t_1, t_2)$ defined as the probability that the process $X(t)$, adjusted to have zero mean, has not changed sign in the interval $[t_1, t_2]$. In several such nonequilibrium systems persistence $p(t_0, t)$, for $t \gg t_0$, decays as a power law, $p_0(t) = p(t_0, t \gg t_0) \sim t^{-\theta}$, with a nontrivial persistence exponent θ [3].

A stochastic process $X(t)$ (depicted schematically in Fig. 1) evidently does not change sign between two consecutive zero crossings. The persistence $p_0(t)$ is simply related to the probability distribution of time intervals (or excursions) between successive zeros and is clearly one, but *not the only one*, possible characterization of the history dependence of $X(t)$. In this Letter we propose an alternative yet simply measurable characteristic of the history of $X(t)$ via an *extreme* observable that elucidates, in a natural way, the important role played by *extreme value statistics* in such nonequilibrium systems. In particular, our results illustrate the *universal* features of extreme statistics in generic many-body nonequilibrium systems and provides, in addition, interesting connections with the theory of records that attracted much interest recently in the context of random walks [5], growing networks [6] and pinned elastic manifolds [7].

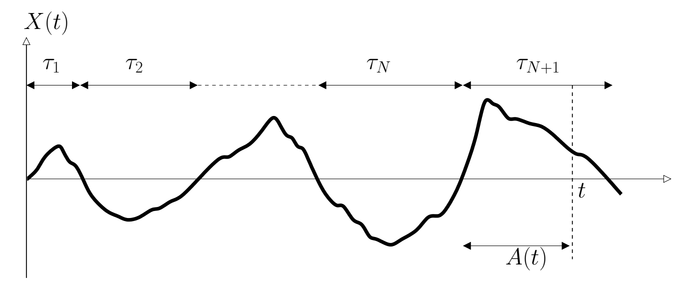

FIG. 1: Intervals between zero-crossings (excursions) of a stochastic process $X(t)$.

For a typical realization of a generic process $X(t)$ with $N \equiv N(t)$ zeros in the *fixed* time interval $[0, t]$ (see Fig. 1), let $\{\tau_1, \tau_2, \cdots, \tau_N\}$ denote the interval lengths between successive zeros and $A(t)$ denote the length (or age) of the last *unfinished* excursion. Our proposed *extreme* observable is the length of the *longest* excursion up to t

$$l_{\max}(t) = \max(\tau_1, \tau_2, \cdots, \tau_N, A(t)) . \tag{1}$$

Let us first summarize our main results. We find rather rich universal late time behavior of the average $\langle l_{\max}(t)\rangle$ for generic stochastic processes $X(t)$ arising in nonequilibrium systems. Such processes typically belong to two broad classes [3]: *smooth* (i.e., with a finite density of zeros) and *non-smooth* (with infinite density of zeros). While persistence typically decays algebraically, $p_0(t) \sim t^{-\theta}$, irrespective of the smoothness of the process, $\langle l_{\max}(t)\rangle$, in contrast, turns out to be sensitive to the smoothness of $X(t)$. For smooth processes, $\langle l_{\max}(t)\rangle$

always grows linearly with time

$$\langle l_{\max}(t)\rangle \simeq Q_\infty \, t \;, \tag{2}$$

albeit with a model dependent prefactor $Q_\infty > 0$. In contrast, for non-smooth processes, it grows either as in (2), or as

$$\langle l_{\max}(t)\rangle \sim t^{1-\psi} \;, \tag{3}$$

where the exponent $0 < \psi < 1$ (sublinear growth), depending on whether the associated persistence exponent θ of the process is less ($\theta < \theta_c$) or greater ($\theta > \theta_c$) than a critical value θ_c. We establish these behaviors via exact analytical calculations for two simple models, one corresponding to each class: multiplicative (smooth) and renewal (non-smooth) processes. In addition, we perform extensive numerical simulations in a variety of nonequilibrium systems, including the diffusion equation with random initial conditions and coarsening dynamics of the Ising model both below and exactly at the critical temperature $T = T_c$. For the latter case, our results (2, 3) provide new universal quantities associated to nonequilibrium critical dynamics.

The full knowledge of the distribution of $l_{\max}$ [8] allows in particular the computation of its average. However, here we compute $\langle l_{\max}(t)\rangle$ by using the relationship

$$d\langle l_{\max}(t)\rangle/dt = Q(t) \;, \tag{4}$$

where $Q(t)$ denotes the probability that the last excursion in $[0,t]$, $A(t)$ in Fig. 1, is the longest one

$$Q(t) = \mathrm{Prob}[l_{\max}(t) = A(t)] \;. \tag{5}$$

Thus $Q(t)$ is the rate at which the "record" length of an excursion is broken at time t. Indeed, if the total interval increases from t to $t+dt$, the random variable $l_{\max}(t)$ either increases by dt (if the last excursion happens to be the longest one and the probability for this event is $Q(t)$) or stays the same (with probability $1 - Q(t)$). Taking average gives (4). Henceforth we focus on $Q(t)$, rather than $\langle l_{\max}(t)\rangle$ directly.

Renewal processes. Let $X(t)$ be a renewal process with successive interval lengths τ_i's being independent random variables, each distributed according to a Lévy law with parameter θ, $\rho(\tau) \sim \tau^{-1-\theta}$, for large τ [9]. The persistence is simply $p_0(t) = \int_t^\infty d\tau \rho(\tau) \sim t^{-\theta}$. The joint distribution $q_N(\tau_1, \tau_2, \cdots, \tau_N, A(t); t)$ of the intervals depicted in Fig. 1 is then

$$\begin{aligned} q_N(\tau_1, \cdots, \tau_N, A(t); t) &= \rho(\tau_1)\rho(\tau_2)\cdots\rho(\tau_N) p_0(A(t)) \\ &\times \delta(\tau_1 + \tau_2 + \cdots + \tau_N + A(t) - t) \;, \end{aligned} \tag{6}$$

where the δ function ensures that the total interval length is t. For the last interval to be the longest, the others must be shorter than it implying that $Q(t)$ in Eq. (5) is

$$Q(t) = \sum_{N=0}^{\infty} \int_0^\infty db \int_0^b d\tau_1 ... \int_0^b d\tau_N \, q_N(\tau_1, \cdots, \tau_N, b; t) \;. \tag{7}$$

Taking Laplace transform of Eq. (7) gives a simple form for $\hat{Q}(s) = \int_0^\infty dt e^{-st} Q(t)$:

$$\hat{Q}(s) = \int_0^\infty db \frac{p_0(b) e^{-sb}}{1 - \int_0^b d\tau \rho(\tau) e^{-s\tau}} \;. \tag{8}$$

Using $\rho(\tau) = -p_0'(\tau)$ in an integration by part, followed by change of variables $b = x/s$ and $\tau = y/s$, lead to an expression convenient for late time asymptotic analysis

$$\hat{Q}(s) = \frac{1}{s} \int_0^\infty dx \frac{p_0(x/s) e^{-x}}{p_0(x/s) e^{-x} + \int_0^x dy \, p_0(y/s) e^{-y}} . \tag{9}$$

For $\theta < 1$, one can take the limit $s \to 0$ directly in Eq. (9), using $p_0(t) \sim (t_0/t)^\theta$ for large t with some non-universal microscopic time scale t_0. Interestingly, t_0 cancels between the numerator and the denominator in Eq. (9), yielding $\hat{Q}(s) \sim Q_\infty^R/s$ and thus $Q(t) \to Q_\infty^R$ for large t (the superscript R refers to *renewal* process), as announced in Eq. (2) with a universal constant that depends only on θ (and not on other details)

$$Q_\infty^R \equiv Q_\infty^R(\theta) = \int_0^\infty \frac{dx}{1 + x^\theta e^x \int_0^x dy \, y^{-\theta} e^{-y}} \;. \tag{10}$$

A special case of this general result, $\theta = 1/2$, corresponds to Brownian motion if one considers only "large" excursions, *i.e.* τ_i's in Fig. 1 larger than some cut-off τ_ϵ. This recovers in a simple way the result $Q_\infty^R(1/2) = 0.626508...$, derived previously by mathematicians [10] using rather complicated, albeit rigorous, method. Note that $Q_\infty^R(\theta)$ in Eq. (10) vanishes as $\theta \to 1$. A plot of $Q_\infty^R(\theta)$ vs. θ is shown in Fig. 2d.

In contrast, for $\theta > 1$, the naive substitution of $p_0(t = y/s) \sim (t_0/t)^\theta$ in the integral in the denominator of Eq. (9) is problematic since the integral diverges. Instead a careful analysis shows that $\int_0^x dy \, p_0(y/s) e^{-y} \propto \langle \tau \rangle \, s$ as $s \to 0$ where $\langle \tau \rangle = \int_0^\infty d\tau \, \tau \rho(\tau)$. This yields, after simple algebra, $\hat{Q}(s) \sim s^{-1/\theta}$ and thus $Q(t) \sim t^{-1+1/\theta}$, as announced in Eq. (3), with

$$\psi = 1 - 1/\theta \;. \tag{11}$$

Thus for renewal processes the change of behavior of $\langle l_{\max}(t)\rangle$ happens at $\theta = \theta_c = 1$. Qualitatively this transition can be understood by simple scaling arguments combining extreme value statistics with the behavior of the sum of independent Lévy variables [8]. The fact that the asymptotics of $\langle l_{\max}(t)\rangle$ for Brownian motion, a highly "non-smooth" process with infinite density of zero-crossings, corresponds to a special case ($\theta = 1/2$) of the renewal process suggests that the latter might qualitatively lead to a good approximation of $\langle l_{\max}(t)\rangle$ for other non-smooth processes such as the coarsening dynamics of the Ising model, and leads us to hypothesise that this change of asymptotic behavior of $\langle l_{\max}(t)\rangle$ at a certain θ_c might be generic for non-smooth processes. Such an approximation of the phase ordering of the Ising model by

a renewal process is also useful for other observables [11]. However, for generic non-smooth processes with $\theta > \theta_c$, the scaling relation $\psi = 1 - 1/\theta$ obtained under renewal approximation is in general not valid and ψ seems to be a new exponent. Numerical results indeed support this hypothesis.

Multiplicative processes. A process $X(t)$ is multiplicative if the locations of its zeros $\{t_1, t_2, \cdots\}$ are such that the successive ratios $U_k = t_{k-1}/t_k$ are independent random variables, each distributed over $U \in [0,1]$ with density $\tilde{\rho}(U)$. While the calculation of $Q(t)$ is difficult for arbitrary $\tilde{\rho}(U)$, it turns out that for the special family of density parametrized by θ, $\tilde{\rho}(U) = \theta U^{\theta-1}$, one can use recent results of Ref. [6] to show that $Q(t) \to Q_\infty^M$ (where M refers to *multiplicative* process), leading to a linear growth of $\langle l_{\max}(t)\rangle$ as in Eq. (2) with

$$Q_\infty^M \equiv Q_\infty^M(\theta) = \int_0^\infty ds\, e^{-s-\theta E(s)} \;, \qquad (12)$$

where $E(s) = \int_s^\infty dx\, e^{-x}/x$. In particular, for uniform distribution, $Q_\infty^M(1) = 0.624329...$, the Golomb-Dickman constant that also describes the asymptotic linear growth of the longest cycle of a random permutation [12]. In Fig. 2d we show a plot of $Q_\infty^M(\theta)$. At variance with renewal processes, $\langle l_{\max}(t)\rangle \simeq Q_\infty^M(\theta)\, t$ for all θ.

To appreciate this result in a more general context, we note that a multiplicative process $X(t)$ is non-stationary by construction. However, when plotted as a function of $T = \ln(t)$, the process becomes a stationary renewal process in T since the successive intervals on the T axis $T_k - T_{k-1}$ become statistically independent. Similarly, it turns out that for many nonequilibrium processes (e.g, diffusion equation with random initial condition), the original non-stationary process in real time t becomes stationary in $T = \ln(t)$ [3] and then the renewal approximation is precisely equivalent to the Independent Interval Approximation (IIA) [13], known to be a very good one for smooth processes [3]. For such smooth processes then, multiplicative process is a good approximation in real time t. Within the IIA, the interval distribution in log-time T decays as $\sim \exp[-\theta T]$ for large T where θ is the associated persistence exponent. In real time t, this then corresponds to a multiplicative process with $\tilde{\rho}(U) \sim \theta U^{\theta-1}$ for small U. If one assumes further that this power law form of $\tilde{\rho}(U)$ holds over the full range of $U \in [0,1]$ one arrives precisely at the model studied above with the parameter θ being the persistence exponent. Thus, the multiplicative process with $\tilde{\rho}(U) = \theta U^{\theta-1}$ seems to be qualitatively a good representative of generic smooth processes, leading to the hypothesis of the asymptotic linear growth of $\langle l_{\max}(t)\rangle$ for such smooth processes. This hypothesis is supported by numerical simulations.

Numerical results. We have computed $Q(t)$ for various processes for which the persistence exponent θ is known either exactly or numerically. Guided by our analytical results, we have considered both non-smooth and smooth processes and the numerical results are consistent with the two broad behaviors announced in Eqs. (2) and (3). In the first case, $Q(t) \to Q_\infty$ as in Eq. (2), as shown in Fig. 2a while in the second case, $Q(t) \sim t^{-\psi}$ as in Eq. (3), as shown in Fig. 2b.

As a prototype of non-smooth processes, we have studied the magnetization in the coarsening dynamics of a d-dimensional ferromagnetic Ising system of linear size L consisting of L^d spins $\sigma_i = \pm 1$, with periodic boundary conditions (pbc). Starting from a random initial condition, the spins evolve via Glauber dynamics with nearest neighbour Ising Hamiltonian $H_{\rm Ising} = -\sum_{\langle i,j\rangle} \sigma_i\sigma_j$. Our results are summarized below:

- In $d = 1$ and at zero temperature, we have computed $Q(t)$ for the local magnetization $X(t) = \sigma_i(t)$, for which $\theta = 3/8$ [14]. Fig. 2a shows a plot of $Q(t)$ vs. t for $L = 128$ and 256. These data show that $Q(t) \to Q_\infty$ with $Q_\infty = 0.725(5)$, which is very close to the analytical value obtained for a renewal process in Eq. (10) with $\theta = 3/8$, for which $Q_\infty^R(3/8) = 0.726531..$ (while for a multiplicative process one has $Q_\infty^M(3/8) = 0.80338...$), see Fig. 2c.
- We obtained a similar behavior, i.e. $Q(t) \to Q_\infty$ for the global magnetization $M(t) = L^{-d}\sum_i \sigma_i(t)$ both in $d = 1$ at $T = 0$ (for which $\theta = 1/4$ [15]) and in $d = 2$ at the critical point $T = T_c$ (for which $\theta = 0.237(3)$ [15]). These results are shown in Fig. 2c. Note that for the global magnetization, the agreement between the numerical value of Q_∞ and the corresponding $Q_\infty^R(\theta)$ is only qualitative.
- The exact solution obtained for renewal processes shows that if θ is large enough (in that case $\theta > 1$), $Q(t)$ decays to zero as a power law $Q(t) \sim t^{-\psi}$ (3, 11). Recently, it was shown that for critical dynamics of Ising systems starting from a completely ordered state, the persistence exponent associated to $X(t) = M(t) - \langle M(t)\rangle$ where $M(t)$ is the global magnetization can be large, for instance $\theta = 1.7(1)$ in $2d$ [16]. Unfortunately, the numerical computation of $Q(t)$ is quite difficult in that case because the exponent ψ is seemingly positive but very small. Alternatively, starting from a fully magnetized state, one can instead consider, as in Ref. [17], the process $X(t) = M_l(t) - \langle M(t)\rangle$ where $M_l(t)$ is the magnetization of a *line*, for which the persistence exponent is even larger $\theta \simeq 3.3$ [18]. In Fig. 2b, we show a plot of $Q(t)$ for this process for two different system sizes $L = 64, 128$. This plot is compatible with a power law decay $Q(t) \sim t^{-\psi}$ with $\psi = 0.34(1)$, which is actually rather far from the value obtained for a renewal process in Eq. (11) which gives $\psi = 1 - 1/\theta \simeq 0.7$.

These results for non-smooth processes in coarsening dynamics, summarized in Fig. 2c, are qualitatively and in some cases even quantitatively (see Fig. 2a) in agreement with the results for renewal processes in Eqs. (10, 11).

As a prototype of smooth processes, we have stud-

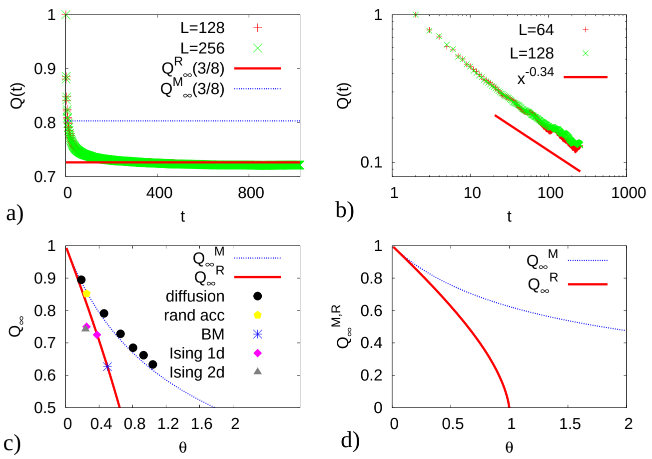

FIG. 2: **a):** $Q(t)$ as a function of t for the *local* magnetization in the Ising chain evolving with Glauber dynamics at $T=0$. **b):** $Q(t)$ as a function of t on a log-log scale for the magnetization of a line in the $2d$-Ising model at T_c evolving with Glauber dynamics starting from a fully magnetized state. **c):** Q_∞ as a function of θ: the lines correspond respectively to $Q_\infty^R(\theta)$ (solid) and $Q_\infty^M(\theta)$ (dotted) and the points correspond to the numerical values obtained for different nonequilibrium systems. The two values for 'Ising 1d' correspond to the local magnetization ($\theta = 3/8$) and to the global one ($\theta = 1/4$). **d)**: Same plot as in c) on a larger scale.

ied the diffusing field $\varphi(\mathbf{x},t)$ evolving according to the heat equation $\partial_t \varphi(\mathbf{x},t) = \nabla^2 \varphi(\mathbf{x},t)$ with pbc in dimension d starting from random initial condition $\langle \varphi(\mathbf{x},t=0)\varphi(\mathbf{x}',t=0)\rangle = \delta^d(\mathbf{x}-\mathbf{x}')$. It is known that the persistence exponent $\theta \equiv \theta(d)$ associated to the diffusing field at the origin $X(t) = \varphi(\mathbf{x}=\mathbf{0},t)$ depends continuously on d [13] and in particular $\theta(d=46) \simeq 1$ [19]. The probability $Q(t)$ can be easily computed numerically in any dimension d by solving the heat equation and noticing that the field at the origin $\varphi(\mathbf{0},t)$ can be simply written, for a large system size, as $\varphi(\mathbf{0},t) \sim \int_0^\infty dr\, r^{(d-1)/2} e^{-r^2/t}\Psi(r)$ where $\Psi(r)$ is a random field with short range correlations. We have computed $Q(t)$ for $d = 2, 10, 20, 30, 40$ and 50 and found, in all cases, $Q(t) \to Q_\infty$. The asymptotic values Q_∞ as a function of θ, reported in Fig. 2c, are in good agreement, even quantitatively, with $Q^M(\theta)$ for multiplicative processes. In Fig. 2c we have also reported the value of Q_∞ for another smooth process called the random acceleration process, for which $\theta = 1/4$. These data suggest that smooth processes, at variance with coarsening dynamics in Ising systems, are better approximated by multiplicative processes.

A close look at Fig. 1 suggests investigation of other closely related cousins of $l_{\max}(t)$ defined in Eq. (1), such as $\mu_{\max}(t) = \max(\tau_1, \tau_2, \cdots, \tau_{N+1})$ or for instance $\lambda_{\max}(t) = \max(\tau_1, \tau_2, \cdots, \tau_N)$. For renewal processes with Lévy index $\theta < 1$, the analysis presented above can be extended to obtain exact results for the average of both observables. In both cases, the average grows linearly with t but with different θ-dependent prefactors. While the prefactor for the former case was computed in Ref. [20] by a rather complicated but rigorous method, the latter case $\langle \lambda_{\max}(t)\rangle$ has not been studied, to our knowledge, even for Brownian motion ($\theta = 1/2$). We find $\langle \lambda_{\max}(t)\rangle \simeq \lambda_\infty(\theta)\, t$ which yields, in particular, a new constant $\lambda_\infty(1/2) = 0.241749...$ for Brownian motion. The detailed studies of $\mu_{\max}(t)$ and $\lambda_{\max}(t)$ for other nonequilibrium processes will be reported elsewhere [8].

In conclusion, we have shown that the average length of the longest excursion has rather rich and universal asymptotic time dependence for a variety of nonequilibrium processes. Our analytical and numerical results highlight the importance of extreme value statistics in generic nonequilibrium dynamics.

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